

MOREIRA'S THEOREM ON THE ARITHMETIC SUM OF DYNAMICALLY DEFINED CANTOR SETS

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ABSTRACT. We present a complete proof of a theorem of C.G. Moreira. Under mild checkable conditions, the theorem asserts that the Hausdorff dimension of the arithmetic sum of two dynamically defined Cantor subsets of the real line, equals either the sum of the dimensions or 1, whichever is smaller.

1. INTRODUCTION

A classical problem in geometric measure theory is to compute or estimate the fractal dimension of the arithmetic sum $K^1 + K^2$ in terms of the dimensions of $K^1, K^2 \subset \mathbb{R}^n$. The sumset $K^1 + K^2$ is, up to affine equivalence, the orthogonal projection of the product set $K^1 \times K^2$ onto the line $\{(t, t)\}_{t \in \mathbb{R}}$, so this problem is related to questions on orthogonal projections. See [PS] for further discussion on this connection and the history of the problem.

Since orthogonal projections are Lipschitz maps and do not increase dimension, it follows that

$$\dim_H(K^1 + K^2) \leq \dim_H(K^1 \times K^2) \leq \overline{\dim}_B(K^1) + \dim_H(K^2),$$

where \dim_H denotes Hausdorff dimension and $\overline{\dim}_B$ denotes upper box (Minkowski) dimension; see [Mat95] for the right-hand side inequality. If $\dim_H(K^1) = \dim_B(K^1)$ (which is the case if, for example, K^1 is the attractor of a self-conformal iterated function system), we obtain the inequality

$$\dim_H(K^1 + K^2) \leq \min(\dim_H(K^1) + \dim_H(K^2), n), \quad (1)$$

where n is the dimension of the ambient space. Obtaining lower bounds is much harder, and it is easy to construct examples where (1) fails. However, there is a heuristic principle which says that “generically” (1) should hold as equality. In some cases this has been accomplished in a measure-theoretical sense, see for example [PS98]. However, from those results one cannot tell whether equality in (1) holds for a *specific* pair K^1 and K^2 .

An **iterated function system** (or i.f.s. for short) is a finite family $\{f_1, \dots, f_m\}$ of self-maps of \mathbb{R}^n (or a more general complete metric space, but here we will only consider iterated function systems on the real line), such that each map f_i is Lipschitz with Lipschitz constant strictly less than 1; in other words, such that

$$|f_i(x) - f_i(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}^n, i \in \{1, \dots, m\},$$

for some constant $L < 1$. If all the maps f_i are C^α for some $\alpha \geq 1$ we will say that the i.f.s. is C^α . For fixed α and m , the family of all C^α iterated function systems with m maps inherits a natural topology from $C^\alpha \times \cdots \times C^\alpha$, where the product is of course m -fold.

Given an i.f.s. $\mathcal{I} = \{f_1, \dots, f_m\}$, the **attractor** $K = K(\mathcal{I})$ is the only nonempty compact subset of \mathbb{R}^n such that

$$K = \bigcup_{i=1}^m f_i(K).$$

See e.g. [Fal90] for more background on iterated function systems, including the existence and uniqueness of attractors.

A **regular Cantor set** $K \subset \mathbb{R}$ is the attractor of a C^2 iterated function system $\{f_1, \dots, f_m\}$ such that the sets $f_i(I)$ are pairwise disjoint, where I is the convex hull of K , and moreover $f_i : I \rightarrow f_i(I)$ is a diffeomorphism. In the dynamics literature regular Cantor sets are usually defined as repellers of smooth expanding maps. The existence of Markov partitions allows to realize such repellers as attractors of iterated function systems of a more general kind (i.e. “graph directed” ones). In this paper we concentrate on the most basic kind of attractors, but this is just a matter of notational simplicity; both the result and the proof extend in a straightforward way to more general repellers.

Moreira and Yoccoz [MY01] proved a deep result about the arithmetic sum of regular Cantor sets K^1, K^2 when $\dim(K^1) + \dim(K^2) > 1$. They prove that generically (in a topological sense with respect to the C^2 topology) such sumsets contain intervals, settling a conjecture of Jacob Palis [Pal87]. The results in [MY01] have important consequences on the study of homoclinic bifurcations.

In a different direction, Moreira [Mor98] studied some problems in diophantine approximation which also involve sums of Cantor sets. As part of the solution to those problems, Moreira states the following result:

Theorem 1. *Let $\{f_1^i, \dots, f_{m_i}^i\}$, $i = 1, 2$, be a pair of C^2 iterated function systems on \mathbb{R} , and let K^1, K^2 be the attractors. Suppose that the families $\{f_j^i\}_{j=1}^{m_i}$, $i \in \{1, 2\}$ are pairwise disjoint. Assume that the following properties hold:*

- (1) *There are $1 \leq i < j \leq m_1$ and x_0 in K^1 such that*

$$(f_i^1 \circ (f_j^1)^{-1})''(x_0) \neq 0.$$

- (2) *There exist $1 \leq l_i \leq m_i$, $i = 1, 2$, such that if y_i is the fixed point of $f_{l_i}^i$ then*

$$\frac{\log |(f_{l_1}^1)'(y_1)|}{\log |(f_{l_2}^2)'(y_2)|} \notin \mathbb{Q}.$$

Then

$$\dim_H(K^1 + K^2) = \min(\dim_H(K^1) + \dim_H(K^2), 1).$$

The hypotheses in this theorem are generic (the first one is robust and dense in the C^1 topology, while the second holds for almost every parameter in generic parametrized families). Moreover, the hypotheses are explicit and can be checked in specific examples.

Unfortunately, the proof of Theorem 1 which appeared in [Mor98] has some errors. Even though the basic idea is correct, it is far from trivial to recover a complete proof for it, and even the basic ideas may be somewhat obscure for those not familiar with the techniques in [MY01]. C.G. Moreira explained to us the main corrections needed; based on this we were able to reconstruct a complete proof of Theorem 1. The purpose of this note is to write down this proof in detail. One motivation for doing this is that we believe that some of the ideas contained in the proof may find application in other problems in geometric measure theory or dynamics, where current methods only yield almost everywhere or random results.

Several developments took place after a first version of this paper was completed. Moreira informed us that he can now prove Theorem 1 without assuming hypothesis (2). Using some of the ideas presented in this paper, but also substantial new ones, Y. Peres and the author [PS] proved a version of Theorem 1 when the f_j^i are all linear maps; this includes classical examples such like central Cantor sets. We also prove that hypothesis (2) is necessary in this case. Eroğlu [Ero07] investigated the Hausdorff measure of sumsets in the critical dimension. In particular, he proves that in many cases it is zero. Thus we now have a rather complete picture of the size of the arithmetic sum of dynamically-defined Cantor sets in the line, at least in the case where the sum of their dimensions does not exceed one.

2. NOTATION

Let $\mathcal{I} = \{f_1, \dots, f_m\}$ be a C^2 i.f.s. on \mathbb{R} (with $\varepsilon > 0$), such that the basic pieces $f_i(K)$ are pairwise disjoint, where $K = K(\mathcal{I})$ is the attractor. We say that \mathcal{I} is **normalized** if the convex hull of K is the unit interval $I = [0, 1]$.

Orientation-preserving (surjective) diffeomorphisms of the unit interval will play an important role. The set of all such diffeomorphisms of class C^1 , endowed with the C^1 topology, will be denoted by \mathcal{G} . An alternative way of thinking of \mathcal{G} is as the space of all diffeomorphic embeddings of the unit interval into \mathbb{R} , divided by the action of the affine group by left composition (the equivalence of both definitions is given by the choice of a representative in a canonical way).

We will use the following form of the C^1 norm:

$$\|f\|_{C^1} = \max\{\|f\|_{L^\infty}, \|f'\|_{L^\infty}\}.$$

. In addition, we let

$$\mathcal{G}(\delta) = \{g \in \mathcal{G} : \|g - \text{Id}\|_{C^1} < \delta\}.$$

We record the following immediate lemma for later reference:

Lemma 1. *Let $f : J \rightarrow \mathbb{R}$ be a diffeomorphism, where J is a closed subinterval of I . Then $\|f - \text{Id}\|_{C^1} = \|f' - 1\|_{L^\infty}$.*

Proof. Obviously $\|f - \text{Id}\|_{C^1} \geq \|f' - 1\|_{L^\infty}$. The other inequality also follows since for $x \in I$ we have

$$|f(x) - x| \leq \int_J |f'(x) - 1| dx \leq \|f' - 1\|_{L^\infty}.$$

□

Let \mathcal{I} be any regular normalized i.f.s. with attractor K . The symbolic space is $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$, where m is the number of maps in the i.f.s. The set of all finite words with symbols in $\{1, \dots, m\}$ will be denoted by Σ^* .

If $u = (u_1, \dots, u_j) \in \Sigma^*$, we will write $f_u = f_{u_1} \circ \dots \circ f_{u_j}$. The reverse word (u_j, \dots, u_1) will be denoted by u^* . We will also let T_u be the unique affine map such that $T_u \circ f_u \in \mathcal{G}$.

Given $\omega \in \Sigma$, let $\omega|k$ be the restriction of ω to the first k coordinates. For a given $\omega \in \Sigma$, consider the sequence $T_{(\omega|k)^*} \circ f_{(\omega|k)^*} \in \mathcal{G}$. Sullivan [Sul88] proved that this sequence converges in C^1 , uniformly in ω (in fact the convergence is in any smoothness class to which the f_i belong, but for us C^1 suffices). The limiting diffeomorphism will be denoted by L_ω , and the image set $L_\omega(K)$ will be called a **limit geometry** of K . Limit geometries are also regular, normalized Cantor sets with the same dimension as K (indeed, $L_\omega K$ is the attractor of $\{L_\omega f_i L_\omega^{-1}\}_{i=1}^m$). Moreover, since the convergence in Sullivan's Theorem is uniform and Σ is compact, the family $\{L_\omega : \omega \in \Sigma\}$ is also compact.

We will consider pairs of iterated function systems $\mathcal{I}^1, \mathcal{I}^2$ on \mathbb{R} , and the product attractor $\Lambda = K^1 \times K^2 = K(\mathcal{I}^1) \times K(\mathcal{I}^2)$. Throughout the paper we will distinguish the i.f.s. we are referring to by the use of a superscript. For example, Σ^1, Σ^2 will denote the symbol spaces corresponding to the i.f.s. $\mathcal{I}^1, \mathcal{I}^2$ respectively. It should be clear from the context whether a superscript is used in this fashion, or to denote a power operation.

We will always denote $d^i = \dim(K^i)$, $i = 1, 2$, and $d = d^1 + d^2$. The images $f_u^i(K^i)$, where $u \in \Sigma^i$, will be referred to as **cylinder sets**, and denoted by $K^i(u)$. We will deal with the convex hull of cylinder sets rather often; the convex hull of $K^i(u)$ will be denoted by $I^i(u)$.

Let $\rho > 0$ be a small number. The **ρ -decomposition** of Λ , denoted by $\Lambda(\rho)$, is the collection of all pairs of words (u_1, u_2) such that

$$|I^i(u_i)| = \text{diam}(K^i(u_i)) > \rho \quad (i = 1, 2),$$

but these inequalities fail for any words containing u_1, u_2 as proper initial subwords. For $\phi = (\phi_1, \phi_2) \in \mathcal{G} \times \mathcal{G}$ we will also abbreviate

$$Q^\phi(u_1, u_2) = \phi_1(I^1(u_1)) \times \phi_2(I^2(u_2)).$$

When $\phi = \text{Id} \times \text{Id}$ (where Id is the identity map) we will simply write $Q(u_1, u_2)$.

Many calculations will depend on a previously fixed constant A . Given two positive quantities x, y , by $x \lesssim y$ we will mean $x < Cy$ for some constant C which depends continuously on A , \mathcal{I}^1 , and \mathcal{I}^2 . We define $x \gtrsim y, x \approx y$ analogously.

When we need to refer to constants explicitly we will denote them by c or C ; their value can be different at each line, and they always depend continuously on A , \mathcal{I}^1 , and \mathcal{I}^2 .

Let $\Pi_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection-type mapping $\Pi(x, y) = x + \lambda y$. Let \mathcal{R} be a subset of the ρ -decomposition $\Lambda(\rho)$. We will say that \mathcal{R} is (η, λ, ϕ) -**faithful** if it contains a subfamily \mathcal{R}' , with $\#\mathcal{R}' > \rho^{\eta-d}$, and such that

$$\{\Pi_\lambda(Q^\phi(u_1, u_2)) : (u_1, u_2) \in \mathcal{R}'\}$$

is a pairwise disjoint family. The following lemma, although very simple, will play a crucial role in the proof:

Lemma 2. *Fix $C_0, \eta > 0$. Then for all sufficiently small ρ (depending on C_0 and η) the following holds: if a family $\mathcal{R} \subset \Lambda(\rho)$ is (η, λ, ϕ) -faithful, it is also $(2\eta, \tilde{\lambda}, \tilde{\phi})$ -faithful for all $\tilde{\lambda}, \tilde{\phi}$ such that*

$$|\tilde{\lambda} - \lambda| \leq C_0\rho, \quad \|\tilde{\phi}_i - \phi_i\|_{C^1} \leq C_0\rho \quad (i = 1, 2).$$

Proof. Note that there exists $C = C(C_0, \eta) > 0$ such that

$$\Pi_{\tilde{\lambda}}\left(Q^{\tilde{\phi}}(u_1, u_2)\right) \subset C \cdot \Pi_\lambda(Q^\phi(u_1, u_2)),$$

where $C \cdot J$ denotes the interval with the same center as J and length $C|J|$. Therefore if \mathcal{R}_1 is the family arising from the definition of (η, λ, ϕ) -faithful, there is a subset \mathcal{R}_2 of \mathcal{R}_1 of cardinality at least $|\mathcal{R}_1|/(2C)$ such that

$$\{C \cdot \Pi_\lambda(Q(u_1, u_2)) : (u_1, u_2) \in \mathcal{R}_2\}$$

is a disjoint family. Taking ρ small enough so that $\rho^\eta < (2C)^{-1}$ yields the lemma. \square

Finally, we define the renormalization operators. This is a family of operators $\{R_{u_1, u_2} : u_i \in (\Sigma^*)^i\}$, defined as

$$R_{u_1, u_2}(\omega_1, \omega_2, s) = \left(u_1\omega_2, u_2\omega_2, |I^1(u_1)|^{-1} |I^2(u_2)| s\right).$$

To understand the action of these operators, define

$$\Lambda_{\omega_1, \omega_2} = L_{\omega_1}(K^1) \times L_{\omega_2}(K^2).$$

We will also refer to $\Lambda_{\omega_1, \omega_2}$ as a limit geometry of Λ . Small cylinders are very close, after rescaling, to a limit geometry, and since we will deal with robust properties (i.e. properties which are invariant under small perturbations of the parameters) we will be able to draw conclusions about limit geometries from its finite approximations.

Moreover, a cylinder of a limit geometry is also close to a corresponding cylinder in the original attractor. In this sense, the action of the renormalization operator R_{u_1, u_2} is to “zoom into” the (u_1, u_2) -cylinder of the given

limit geometry (or approximating cylinder); the transformation of s simply takes into account the normalization (rescaling back to the unit square) of the cylinder.

3. AUXILIARY RESULTS

In this section we collect a number of basic results that we will use in the course of the proof of the main theorem.

Lemma 3. *Let $\mathcal{I} = \{f_i\}_{i=1}^m$ be a regular i.f.s. with attractor K . Then the following holds:*

- (i) *If f_1 is linear then for all finite words u the limit geometry $L_{u1^\infty}(K)$ is affinely equivalent to the cylinder $f_{u^\star}(K)$. More precisely, we have the identity*

$$L_{u1^\infty} = T_{u^\star} f_{u^\star}.$$

- (ii) *The map $g_j = L_{j^\infty} f_j L_{j^\infty}^{-1}$ is linear, and its eigenvalue is equal to the eigenvalue of f_j .*
- (iii) *Suppose that f_1 is linear, and fix a finite word u with symbols in $\{1, \dots, m\}$. Consider a new i.f.s $\{h_1, \dots, h_m\}$, where*

$$h_i = L_{u1^\infty} f_i L_{u1^\infty}^{-1}.$$

Then for all words v ,

$$|h_{v^\star}(I)| = \frac{|I((vu)^\star)|}{|I(u^\star)|}.$$

Proof.

- (i) This is obvious when thinking of \mathcal{G} as a quotient space.
- (ii) Again using the quotient space interpretation, it is clear that the class of $L_{j^\infty} f_j$ is the same as the class of L_{j^∞} , and therefore the class of $L_{j^\infty} f_j L_{j^\infty}^{-1}$ is the affine group. The invariance of eigenvalues under conjugacies is a general fact.
- (iii) Keeping in mind that T_z is linear and using (i) we have:

$$\begin{aligned} |h_{v^\star}(I)| &= |T_{u^\star} f_{u^\star} f_{v^\star} (T_{u^\star} f_{u^\star})^{-1} I| \\ &= |T_{u^\star} f_{u^\star} f_{v^\star} I| = |T_{u^\star} f_{(vu)^\star} I| \\ &= |T_{u^\star} I| |T_{(vu)^\star} I|^{-1} = |I(u^\star)|^{-1} |I((vu)^\star)|. \end{aligned}$$

□

The previous lemma will allow us to assume that the maps in K^1, K^2 for which the incommensurability holds are actually linear, and this in turn will imply that limit geometries are cylinder sets.

We will need the well-known **bounded distortion principle** (see e.g. [Fal97, Proposition 4.2]) :

Lemma 4. *Let \mathcal{I} be a $C^{1+\varepsilon}$ i.f.s. for some $\varepsilon > 0$, and let K be the attractor. Then there is $L_1 > 0$ such that*

$$L_1^{-1} < \frac{|f'_u(x)|}{|f'_u(y)|} < L_1,$$

for all $u \in \Sigma$ and all $x, y \in K$.

The following proposition is a key geometrical result. It is a kind of discrete Marstrand theorem on projections in a particular setting (see [Mat95, Chapter 10] for general projection theorems). Intersection numbers are used in a similar fashion in the work of M. Rams, see [Ram05] and references therein. In [PS] a general discrete projection theorem is proved, but the result we need here does not follow directly from it, so a full proof is given for the convenience of the reader.

Proposition 5. *Fix a large constant A and a small constant $\eta > 0$. Let K^1, K^2 be attractors of regular normalized i.f.s. $\mathcal{I}^1, \mathcal{I}^2$ respectively, $d^i = \dim_H(K^i)$, $\Lambda = K^1 \times K^2$ and $d = d^1 + d^2 = \dim_H(\Lambda)$. Assume that $d < 1$.*

Then there is a number $\rho_0 > 0$, which depends continuously on η, A, \mathcal{I}^1 and \mathcal{I}^2 , such that for all $0 < \rho < \rho_0$ there exists a set $J \subset [-A, A]$ with the following properties:

- (1) $\mathcal{L}([-A, A] \setminus J) < \rho^\eta$, where \mathcal{L} denotes one-dimensional Lebesgue measure.
- (2) If $\lambda \in J$ and \mathcal{R} is any subset of the ρ -decomposition such that $\#\mathcal{R} > \rho^{\eta-d}$, then \mathcal{R} is $(4\eta, \lambda, \text{Id})$ -faithful.

Proof. Let

$$N(\lambda) = \#\{(u, v) \in \Lambda(\rho) \times \Lambda(\rho) : \Pi_\lambda(Q(u)) \cap \Pi_\lambda(Q(v)) \neq \emptyset\}.$$

Note that in the above u and v are pairs of words. Given $u, v \in \Lambda(\rho)$ let also

$$E(u, v) = \{\lambda : \Pi_\lambda(Q(u)) \cap \Pi_\lambda(Q(v)) \neq \emptyset\}.$$

Observe that if $\lambda \in E(u, v)$ then there is a line with slope λ intersecting both $Q(u)$ and $Q(v)$. Therefore we have the estimate

$$\mathcal{L}(E(u, v)) \lesssim \rho / \text{dist}(Q(u), Q(v)). \quad (2)$$

As a consequence of the bounded distortion principle, the following holds: given $u \in \Lambda(\rho)$ and $\varepsilon > \rho$,

$$\#\{v \in \Lambda(\rho) : \text{dist}(Q(u), Q(v)) < \varepsilon\} \lesssim (\rho/\varepsilon)^{-d}. \quad (3)$$

The constant implied by the \lesssim notation depends continuously on $\mathcal{I}^1, \mathcal{I}^2$. In particular, $\#\Lambda(\rho) \lesssim \rho^{-d}$, and $\text{dist}(Q(u), Q(v)) \gtrsim \rho$ for any two different u, v in the ρ -decomposition.

Claim.

$$\int_{-A}^A N(\lambda) d\lambda \lesssim \rho^{-d}.$$

Proof of Claim. Given $u, v \in \Lambda(\rho)$ let $d(u, v) = \text{dist}(Q(u), Q(v))$. We estimate:

$$\begin{aligned}
\int_{-A}^A N(\lambda) d\lambda &= \sum_{u \in \Lambda(\rho)} \sum_{v \in \Lambda(\rho)} \int_{-A}^A \mathbf{1}_{\{\Pi_\lambda(Q(u)) \cap \Pi_\lambda(Q(v)) \neq \emptyset\}} d\lambda \\
&= \sum_{u \in \Lambda(\rho)} \sum_{i=0}^{-\log_2(C\rho)} \sum_{1 < 2^i d(u, v) \leq 2} \mathcal{L}(E(u, v)) \\
&\lesssim \rho \sum_{u \in \Lambda(\rho)} \sum_{i=0}^{-\log_2(C\rho)} 2^i \#\{v \in \Lambda(\rho) : d(u, v) < 2^{1-i}\} \\
&\lesssim \rho \sum_{u \in \Lambda(\rho)} \sum_{i=0}^{-\log_2(C\rho)} 2^i (\rho 2^{i-1})^{-d} \\
&\lesssim \rho^{1-d} \#\Lambda(\rho) 2^{-(1-d) \log_2(C\rho)} \\
&\approx \rho^{-d},
\end{aligned}$$

where we used (2) in the third line and (3) in the fourth line. This proves the claim. Let J be defined as

$$J = \{\lambda \in [-A, A] : N(\lambda) < \rho^{-2\eta-d}\}.$$

We will show that J has the desired properties. Firstly, by the claim and Chebychev's inequality,

$$\mathcal{L}([-A, A] \setminus J) \lesssim \rho^{2\eta} \implies \mathcal{L}([-A, A] \setminus J) < \rho^\eta,$$

if ρ is small enough. Now let \mathcal{R} be a subset of $\Lambda(\rho)$ such that $\#\mathcal{R} > \rho^{\eta-d}$, and define

$$N_1(\lambda) = \#\{(u, v) \in \mathcal{R} \times \mathcal{R} : \Pi_\lambda(Q(u)) \cap \Pi_\lambda(Q(v)) \neq \emptyset\},$$

and define J_1 analogously using N_1 instead of N . Clearly $N_1(\lambda) \leq N(\lambda)$ for all λ , whence $J \subset J_1$. Therefore it is enough to prove that condition (2) applied to \mathcal{R} in the proposition holds for any fixed $\lambda \in J_1$.

Note that if $\lambda \in [-A, A]$ then $\Pi_\lambda(\Lambda) \subset [-A-1, A+1]$. Let us divide $[-A-1, A+1]$ into intervals I_j of length slightly less than ρ , $j = 1, \dots, \lceil (2A+2)\rho^{-1} \rceil$. Write m_j for the number of rectangles $Q = Q(u_1, u_2)$, where $(u_1, u_2) \in \mathcal{R}$, such that the center of I_j belongs to $\Pi_\lambda(Q)$. Also let

$$\mathcal{J} = \{1 \leq j \leq \lceil (2A+2)\rho^{-1} \rceil : m_j > 0\},$$

and observe that it is enough to show that $\#\mathcal{J} \gtrsim \rho^{4\eta-d}$ (provided this holds, for each $j \in \mathcal{J}$ we pick $(u_1, u_2) \in \mathcal{R}$ such that the center of I_j belongs to $\Pi_\lambda(Q(u_1, u_2))$; by construction this is a family with a bounded covering number, so we can pick an appropriate disjoint subfamily \mathcal{R}' with comparable cardinality).

Note that each $\Pi_\lambda(Q)$ contains the center of a uniformly bounded number of I_j , and therefore

$$\sum_{j \in \mathcal{J}} m_j \gtrsim \#\mathcal{R} \geq \rho^{\eta-d}.$$

Using this we estimate, for sufficiently small ρ ,

$$\begin{aligned} \rho^{-2\eta-d} &> N_1(\lambda) \geq \sum_{j \in \mathcal{J}} m_j^2 \\ &\geq (\#\mathcal{J})^{-1} \left(\sum_{j \in \mathcal{J}} m_j \right)^2 \geq (\#\mathcal{J})^{-1} \rho^{2\eta-2d}. \end{aligned}$$

This concludes the proof of the proposition. \square

We remark that because of the compactness of the set of limit geometries, the number ρ_0 given by the proposition can be chosen uniformly for all limit geometries $\Lambda_{\omega_1, \omega_2}$.

4. PROOF OF THE MAIN THEOREM

4.1. Sketch of proof. We begin by sketching the proof; full details follow below. The bulk of the proof consists in showing that given $\varepsilon > 0$, the inequality

$$\dim_H(\Pi_\lambda(\Lambda)) > d - \varepsilon$$

holds for λ in some *open* set. Moreover, to begin with we can assume that \mathcal{I}^1 and \mathcal{I}^2 both contain a linear map. From here one can deduce, using incommensurability, that the same inequality holds for *all* $\lambda \neq 0, \infty$, and then pass to the general case by approximating cylinders by limit geometries and using Lemma 3.

We fix a small η and for each $\rho > 0$ apply Proposition 5 to obtain many (more precisely, all up to a small exponential correction) rectangles in the ρ -decomposition with disjoint projections. For a fixed ρ , this construction is robust in λ (perturbing the λ slightly the rectangles will have projections at a distance of at least, say, $\rho/2$).

The goal is to carry this construction inductively in each of those rectangles, but *a priori* there is a big obstacle: the set of parameters λ given by Proposition 5, even though of almost full measure, can vary for each rectangle, so we need some device to make sure that the parameters are *recurrent*; i.e. we can take the *same* set J for all rectangles, perhaps at the price of reducing the number of rectangles we are working with slightly. Such recurrence result was obtained in [MY01] and is one of the main technical tools in the proof of Theorem 1.

This inductive construction yields, for each λ in some open set, a Moran construction whose limit set is contained in $\Pi_\lambda(\Lambda)$ and has dimension at least $d - \varepsilon$, provided η and then ρ were taken sufficiently small. This finishes the sketch of the proof.

4.2. The scale recurrence lemma. For the convenience of the reader we state the key scale recurrence lemma (sometimes called the scale selection lemma). For the proof, the reader is referred to [MY01].

Let us say that a regular i.f.s. \mathcal{I} is **essentially nonlinear** if it verifies condition (1) in Theorem 1; in other words, if there exist $i < j, x_0 \in K$ such that

$$(f_i \circ f_j^{-1})''(x_0) \neq 0.$$

Theorem 2. *Given regular Cantor sets K^1, K^2 , such that at least one of K^1, K^2 is essentially nonlinear, there exists a large constant A such that, setting $a = A^{-1}$, the following holds:*

Let ρ be sufficiently small. Suppose that for each $\omega_1 \in \Sigma^1, \omega_2 \in \Sigma^2$ some measurable set J_{ω_1, ω_2} is given such that

$$\mathcal{L}(I_A \setminus J_{\omega_1, \omega_2}) < a,$$

where $I_A = [-A, -1/A] \cup [1/A, A]$. Then there exists another family

$$\{F_{\omega_1, \omega_2}\}_{\omega_1 \in \Sigma^1, \omega_2 \in \Sigma^2},$$

verifying the following properties:

- (1) F_{ω_1, ω_2} is contained in the $(A\rho)$ -neighborhood of J_{ω_1, ω_2} .
- (2) For every $s \in F_{\omega_1, \omega_2}$ there are at least $a\rho^{-d}$ elements of the ρ -decomposition of $\Lambda_{\omega_1, \omega_2}$ such that if (u_1, u_2) is one such element and

$$R_{u_1, u_2}(\omega_1, \omega_2, s) = (u_1\omega_1, u_2\omega_2, s'),$$

then $(s' - \rho, s' + \rho) \in F_{u_1\omega_1, u_2\omega_2}$.

4.3. The core of the proof. We now start the proof of Theorem 1. We start by proving a weaker result; Theorem 1 will be obtained later as a corollary.

Proposition 6. *Let K^1, K^2 be regular Cantor sets of dimension d^1, d^2 , such that $d^1 + d^2 < 1$ and K^1 is essentially nonlinear. Assume also that the maps $f_1^i, i = 1, 2$, are linear.*

Then for all $\varepsilon > 0$ there exist nonempty open sets $U^+ \subset \mathbb{R}^+, U^- \subset \mathbb{R}^-$ and $\delta > 0$ such that

$$\dim_H(\phi_1(K^1) + \lambda\phi_2(K^2)) > d^1 + d^2 - \varepsilon \quad (4)$$

for all $\lambda \in U^+ \cup U^-$ and all $\phi_1, \phi_2 \in \mathcal{G}(\delta)$.

Proof. Let A be the constant given by the scale recurrence lemma, and write $a = 1/A$. Fix a small $\eta > 0$, and then a very small $\rho > 0$ so that $\rho^\eta < a$, Proposition 5 works for this ρ for all limit geometries $\Lambda_{\omega_1, \omega_2}$ and the Scale Recurrence Lemma is satisfied. In the course of the proof we will need ρ to satisfy additional conditions; it will be clear that all can be satisfied by starting with a sufficiently small ρ .

For each pair $(\omega_1, \omega_2) \in \Sigma_1 \times \Sigma_2$ let J_{ω_1, ω_2} be the set given by Proposition 5 applied to the limit geometry $\Lambda_{\omega_1, \omega_2}$.

We apply Theorem 2 to obtain a new family $\{F_{\omega_1, \omega_2}\}$ with the conditions prescribed in the scale recurrence lemma. Clearly if ρ is small then F_{ω_1, ω_2} contains both positive and negative numbers. Pick any $\lambda^+ \in F_{1^\infty, 1^\infty} \cap \mathbb{R}^+$. There exists an open set $U = U^+ = (\lambda^+ - c\rho, \lambda^+ + c\rho)$ such that the following holds: for all (u_1, u_2) arising from part (2) of the scale recurrence lemma (applied to $F_{1^\infty, 1^\infty}$, λ^+ , and ρ) and all $\lambda \in U$,

$$|I^1(u_1)|^{-1}|I^2(u_2)|\lambda \in F_{u_1 1^\infty, u_2 1^\infty}. \quad (5)$$

This follows from the fact that the quotients $|I^2(u_2)|/|I^1(u_1)|$ are uniformly bounded. We now fix any $\lambda \in U$ for the rest of the proof (the construction of U^- is exactly analogous). We also fix $\phi = (\phi_1, \phi_2) \in \mathcal{G}(\rho/2) \times \mathcal{G}(\rho/2)$.

We will inductively construct a tree \mathcal{T} , with vertices labeled by pairs of words (u_1, u_2) , such that the following holds: Let \mathcal{T}_k denote the set of vertices of step k .

- (A) If $(u_1, u_2) \in \mathcal{T}_{k+1}$ then $u_i = v_i z_i$ for some (z_1, z_2) , where $(v_1, v_2) \in \mathcal{T}_k$ is the parent of (u_1, u_2) .
- (B) If $(u_1, u_2) \in \mathcal{T}_k$ then $|I^i(u_i)| \geq \rho^k$, $i = 1, 2$. In particular,

$$|\Pi_\lambda(Q^\phi(u_1, u_2))| \gtrsim \rho^k.$$

- (C) Each vertex has $\gtrsim \rho^{9\eta-d}$ offspring.
- (D) For each vertex $(u_1, u_2) \in \mathcal{T}$ the following family is pairwise disjoint:

$$\left\{ \Pi_\lambda(Q^\phi(w_1, w_2)) : (u_1, u_2) \text{ is a parent of } (w_1, w_2) \right\}.$$

Properties (A)-(D) imply that \mathcal{T} induces a separated Moran construction with cylinders $\Pi_\lambda(Q^\phi(u_1, u_2))$, with limit set

$$M = \bigcap_{k=1}^{\infty} \bigcup_{(u_1, u_2) \in \mathcal{T}_k} \Pi_\lambda(Q^\phi(u_1, u_2)).$$

It is clear that $M \subset \Pi_\lambda(\phi_1 K^1 \times \phi_2 K^2)$. Moreover,

$$\dim_H(M) \geq d - 9\eta. \quad (6)$$

This follows by standard methods; we sketch the proof for the convenience of the reader. We construct a probability measure μ supported on M inductively as follows: suppose $\Pi_\lambda(Q^\phi(u_1, u_2))$ has been defined for all $(u_1, u_2) \in \mathcal{T}_k$. Then we distribute the mass of $\Pi_\lambda(Q^\phi(u_1, u_2))$ uniformly among all the offspring intervals $\Pi_\lambda(Q^\phi(v_1, v_2))$ (where $(v_1, v_2) \in \mathcal{T}_{k+1}$ ranges over the offspring of (u_1, u_2)). Using (B), (C) and (D), it is easy to verify that

$$\mu(x - r, x + r) \lesssim r^{d-9\eta},$$

for all $x \in \text{supp}(\mu) = M$ and all $r > 0$. Thus (6) follows from the mass distribution principle (see [Fal97, Proposition 2.1]).

Since η is arbitrary, it will be enough to verify properties (A)-(D) to complete the proof.

For each $(u_1, u_2) \in \mathcal{T}_j$ we will also inductively construct a scale λ^{u_1, u_2} such that $\lambda^{u_1, u_2} \in F_{u_1^* 1^\infty, u_2^* 1^\infty}$ (for $j > 0$). We start by setting $\mathcal{T}_0 = \{(\emptyset, \emptyset)\}$ (the root of the tree; here \emptyset denotes the empty word) and $\lambda^{\emptyset, \emptyset} = \lambda$.

Now we specify the inductive construction: suppose that $(u_1, u_2) \in \mathcal{T}_j$ for some j , and that λ^{u_1, u_2} has been defined. Let $\omega_i = u_i^* 1^\infty$, $i = 1, 2$, and let us apply the scale recurrence lemma to $\Lambda_{\omega_1, \omega_2}$ with scale $s = \lambda^{u_1, u_2}$. We thus obtain a family of pairs of words $\mathcal{R}_0^{u_1, u_2}$ given by the scale recurrence lemma; i.e. $\#\mathcal{R}_0^{u_1, u_2} > a\rho^{-d} > \rho^{\eta-d}$, and if $(v_1^*, v_2^*) \in \mathcal{R}_0^{u_1, u_2}$ and we let

$$\lambda^{v_1 u_1, v_2 u_2} = \lambda^{u_1, u_2} |I_{\omega_1}^1(v_1^*)|^{-1} |I_{\omega_2}^2(v_2^*)|,$$

then $\lambda^{v_1 u_1, v_2 u_2} \in F_{v_1^* u_1^* 1^\infty, v_2^* u_2^* 1^\infty}$; here $I_{\omega_i}^i$ are cylinder intervals with respect to the limit geometries $L_{\omega_i}(K^i)$. For $j = 0$ this follows from (5). From Lemma 3(iii) we get

$$\lambda^{v_1 u_1, v_2 u_2} = \frac{|I(u_1)|}{|I(u_1 v_1)|} \frac{|I(u_2 v_2)|}{|I(u_2)|} \lambda^{u_1, u_2}. \quad (7)$$

We next use Proposition 5, Lemma 2 and the first part of the scale recurrence lemma to obtain a subset $\mathcal{R}_1^{u_1, u_2}$ of $\mathcal{R}_0^{u_1, u_2}$ such that

- (i) $\#\mathcal{R}_1^{u_1, u_2} > \rho^{8\eta-d}$.
- (ii) If $\|\psi_i - \text{Id}\|_{C^1} < \rho$ for $i \in 1, 2$, then

$$\left\{ \Pi_{\lambda^{u_1, u_2}} \left(Q_{u_1^* 1^\infty, u_2^* 1^\infty}^\psi(v_1, v_2) \right) : (v_1^*, v_2^*) \in \mathcal{R}_1^{u_1, u_2} \right\} \quad (8)$$

is a pairwise disjoint family, where $Q_{\omega_1, \omega_2}^\psi$ denotes the rectangle relative to the limit geometry $\Lambda_{\omega_1, \omega_2}$ (or rather the pair of iterated function systems defining it).

We will later construct a family $\mathcal{R}^{u_1, u_2} \subset \mathcal{R}_1^{u_1, u_2}$ such that $\#\mathcal{R}^{u_1, u_2} > \rho^{9\eta-\rho}$. Assuming such a family is given, we define the set of offspring of (u_1, u_2) to be

$$V(u_1, u_2) = \{(u_1 v_1, u_2 v_2) : (v_1^*, v_2^*) \in \mathcal{R}^{u_1, u_2}\}.$$

Properties (A) and (C) of \mathcal{T} are clear from the construction. Property (B) also follows since all $(u_1, u_2) \in \mathcal{T}_k$ are obtained by going to the ρ -decomposition and then rescaling back to the unit square k times. We will now consider property (D); along the way we will define the family \mathcal{R}^{u_1, u_2} precisely.

Notice that from (7) and induction we get that for all k and all $(u_1, u_2) \in \mathcal{T}_k$,

$$\lambda^{u_1, u_2} = \frac{|I(u_2)|}{|I(u_1)|} \lambda. \quad (9)$$

Arguing as in the proof of Lemma 3(iii), we get

$$I_{u_i^* 1^\infty}^i(v_i) = T_{u_i}^i f_{u_i}^i f_{v_i}^i (T_{u_i}^i f_{u_i}^i)^{-1}(I) = T_{u_i}^i f_{u_i v_i}^i(I),$$

whence

$$Q_{u_1^* 1^\infty, u_2^* 1^\infty}^\psi(v_1, v_2) = \psi_1 T_{u_1}^1 I^1(u_1 v_1) \times \psi_2 T_{u_2}^2(I^2(u_2 v_2)).$$

Since the family in (8) is pairwise disjoint, it follows from (9) that (for fixed $(u_1, u_2) \in \mathcal{T}_k$) the family

$$\left\{ |I^1(u_1)|\psi_1 T_{u_1}^1 I^1(u_1 v_1) + \lambda |I^2(u_2)|\psi_2 T_{u_2}^2 (I^2(u_2 v_2)) : (v_1^*, v_2^*) \in \mathcal{R}_1^{u_1, u_2} \right\} \quad (10)$$

is also pairwise disjoint.

For $i \in \{1, 2\}$, let $S_i(x) = \mu_i x + \tau_i$ be the positively-oriented affine map such that $S_i \phi_i$ fixes $I^i(u_i)$. Since $\|\phi_i - \text{Id}\|_{C^1} < \rho/2$, straightforward calculations and Lemma 1 show that $|\mu_i - 1| < \rho/2$ and, restricted to $I^i(u_i)$, $\|S_i \phi_i - \text{Id}\|_{C^1} < \rho$.

Now let

$$\psi_i = T_{u_i}^i S_i \phi_i (T_{u_i}^i)^{-1}.$$

Notice that $\psi_i \in \mathcal{G}$ and, by the previous remarks and Lemma 1, indeed $\psi_i \in \mathcal{G}(\rho)$. Observe also that $\{|I^i(u_i)|T_{u_i}^i\}_{i=1,2}$ are translation maps. We deduce that

$$|I^i(u_i)|\psi_i T_{u_i}^i = |I^i(u_i)|T_{u_i}^i S_i \phi_i = \mu_i \phi_i + \tau'_i,$$

for some $\tau'_i \in \mathbb{R}$. Since affine images of pairwise disjoint families are still pairwise disjoint, we conclude from (10) that the following family is pairwise disjoint as well:

$$\left\{ \phi_1(I^1(u_1 v_1)) + \frac{\lambda \mu_2}{\mu_1} \phi_2(I^2(u_2 v_2)) : (v_1^*, v_2^*) \in \mathcal{R}_1^{u_1, u_2} \right\}.$$

Note that, since $|\mu_i - 1| < \rho/2$,

$$\left| \frac{\mu_2}{\mu_1} - 1 \right| < \frac{\rho}{1 - \rho}.$$

Hence $|\mu_2 \lambda / \mu_1 - \lambda| < 2A\rho$ whenever $\rho < 1/2$, and it follows from Lemma 2 (or its proof) that, provided ρ is small enough, there exists a subfamily $\mathcal{R}^{u_1, u_2} \subset \mathcal{R}_1^{u_1, u_2}$ such that

$$\#\mathcal{R}^{u_1, u_2} \gtrsim \mathcal{R}_1^{u_1, u_2} > \rho^{9\eta-d},$$

and

$$\left\{ \Pi_\lambda Q^\phi(u_1 v_1, u_2 v_2) : (v_1^*, v_2^*) \in \mathcal{R}^{u_1, u_2} \right\} \quad (11)$$

is a pairwise disjoint family. This completes the proof of Proposition 6. \square

4.4. Conclusion of the proof. We now complete the proof of Theorem 1. First of all notice that we can assume that $d < 1$; if $d \geq 1$ just throw away some maps in the first i.f.s. (after a suitable iteration) to obtain a subset of Λ of dimension less than, but arbitrarily close to, 1.

Assume first that f_1^i , $i = 1, 2$, are linear maps, and that $\log r_1 / \log r_2 \notin \mathbb{Q}$, where r_i is the similarity ratio of f_1^i . Fix $\varepsilon > 0$, and let

$$S = \left\{ \lambda : \dim_H(\Pi_\lambda(\phi_1(K^1) \times \phi_2(K^2))) > d - \varepsilon \forall \phi_i \in \mathcal{G}(\delta) \right\}.$$

By Proposition 6, if δ is small enough then S contains some open set U intersecting both the positive and negative half-lines. Now let $k, l \in \mathbb{N}$. The cylinders $K^1(1^k)$, $K^2(1^l)$ are, by hypothesis, affine images of K^i with scaling

factors r_i^k, r_i^l . This implies that S contains the scaling of U by $\pm r_2^l/r_1^k$ (the sign depending on the orientation of the maps f_1^i). But U meets both the positive and negative half-lines, and the set of all such scaling factors is dense by the irrationality assumption, so we conclude that $S = \mathbb{R}^* := \mathbb{R} \setminus \{0\}$.

Next we drop the hypothesis that f_1^1, f_1^2 are linear; we still assume that $\log r_1/\log r_2 \notin \mathbb{Q}$, where r_i is the eigenvalue of f_1^i at its fixed point. By Lemma 3(ii) and the above, there is $\delta > 0$ such that if $\|\phi_i - \text{Id}\|_{C^1} < \delta$ for $i \in \{1, 2\}$ and $\lambda \in \mathbb{R}^*$, then

$$\dim_H(\Pi_\lambda(\phi_1 L_{1^\infty}(K^1) \times \phi_2 L_{1^\infty}(K_2))) > d - \varepsilon.$$

Let k be so large that $\|T_{1^k}^i f_{1^k}^i L_{1^\infty}^{-1}\|_{C^1} < \delta$, $i \in \{1, 2\}$. Then

$$\dim_H\left(\Pi_\lambda\left(T_{1^k}^1 K^1(1^k) \times T_{1^k}^2 K^2(1^k)\right)\right) > d - \varepsilon \text{ for all } \lambda \in \mathbb{R}^*.$$

But since this holds for all $\lambda \neq 0$, $T_{1^k}^i$ is linear and non-degenerate, and $K^i(1^k) \subset K^i$ for $i \in \{1, 2\}$, we obtain that

$$\dim_H(\Pi_\lambda(K^1 \times K^2)) > d - \varepsilon \text{ for all } \lambda \in \mathbb{R}^*.$$

Since ε was arbitrary, this concludes the proof. \square

Acknowledgements. I thank C.G.T. de A. Moreira for explaining the ideas of the proof of Theorem 1 to me, and E. Järvenpää for useful comments and corrections on an early version of the article.

This note was written while I was a postdoc at the University of Jyväskylä. I acknowledge financial support from the Academy of Finland.

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